BMath IInd year
Algebra IV
Supplementary Exam
2015-2016

Full marks 100
Time: 3 hours

Answer all questions.

(1) Show that if $K|F$ is a Galois extension, and $F'|F$ any extension, the compositum extension $KF'|F$ is a Galois extension with Galois group isomorphic to a subgroup of $Gal(K|F)$. (7)

(2) (a) Determine the Galois group of the cyclotomic field $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$, where ζ_n denotes a primitive nth root of unity.
(b) Determine intermediate fields of the extension $\mathbb{Q}(\zeta_7)$ over $\mathbb{Q}$.
(c) Show that each of these intermediate extensions are simple, by finding primitive elements. (5+5+5)

(3) Prove that every finite group occurs as the Galois group of some field extension. (8)

(4) Prove that the polynomial $x^{p^n} - x$ over $\mathbb{F}_p$ is the product of all distinct irreducible polynomials in $\mathbb{F}_p[x]$ of degree d, where d runs over all divisors of n. (10)

(5) (a) Define discriminant D_f of a polynomial $f(x)$.
(b) Let $char(F) \neq 2$ and let $f(x)$ be a separable polynomial of degree n. Show that the Galois group of $f(x)$ over F is a subgroup of the alternating group $A_n(\subset S_n)$ if and only if D_f is a square in F. (3+12)

(6) (a)Define a cyclic extension.
(b) Show that any cyclic extension of degree n over a field F, of characteristic not dividing n, which contains the nth roots of unity, is of the form $F(\sqrt[n]{a})$ for some $a \in F$. (3+12)

(7) (a) Define root extension of a field F.
(b) Let $K|F$ be a root extension. Then show that there exists an extension $L|K$ such that $L|F$ is a Galois root extension. (3+12)

(8) Show that the roots of the polynomial $f(x) = x^5 - 6x + 3$ over $\mathbb{Q}[x]$ cannot be expressed by radicals. (15)

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